JOURNAL OF APPROXIMATION THEORY 18, 108–118 (1976)

An Extension of Saff's Theorem on the Convergence of Interpolating Rational Functions*

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Communicated by Oved Shisha

Received March 10, 1975

In this paper we extend to general interpolation schemes which satisfy only a mild regularity condition, a theorem of Saff on the convergence of interpolating rational functions with a fixed number of poles. The approach is to use the interpolation scheme to define a measure whose logarithmic potential is then shown to be the appropriate majorizing function for establishing convergence.

1. INTRODUCTION

Recently, Saff [6] extended the classical result of Montessus de Ballore on the convergence of Padé fractions with a fixed number of poles to the convergence of interpolating rational functions with a fixed number of poles. However, the hypotheses of Saff's Theorem contain the restriction that the points of the interpolation scheme be properly distributed with respect to a Green's function. In this paper we extend Saff's result to general interpolation schemes which satisfy only a mild regularity condition.

Our approach is to use the interpolation scheme to define a unique measure whose logarithmic potential is the appropriate function for majorizing the remainder term. For interpolation schemes restricted to the real line, this approach has been exploited by Gontscharoff [3] and Krylov [5]. Moreover, it is implicit to some extent in the treatment of the n-point Hermite Interpolation Problem by Davis [1] and Walsh [8]. However, the general case requires some of the results from potential theory. The interpolation measure and its properties are presented in Section 2. The Majorization Theorem is established in Section 3. With this machinery, the extension of Saff's theorem is entirely straightforward, and this extension is developed in Section 4. In Section 5 we illustrate some aspects of the Majorization Theorem with a classical problem of Runge.

* This research was done at the University of California at San Diego and was supported in part by the Air Force Office of Scientific Research under USAF Grant 71-2006.

2. The Interpolation Measure

Let Z denote a bounded triangular interpolation scheme.

$$\begin{array}{lll} z_{00} & & \\ z_{10} & z_{11} & \\ z_{20} & z_{21} & z_{22} \\ \vdots & \vdots & \vdots \end{array}$$

Let E denote the smallest closed (hence compact) set containing the points of Z, and let $M(E)$ denote the set of normalized (i.e., $\mu(E) = 1$), positive Borel measures with support in E.

Definition 1. Given a triangular interpolation scheme, Z, the associated *elementary measures* $\{\mu_n\}$, are defined by

$$\mu_n(B) = \frac{1}{n+1} \sum_{i=0}^{n} \chi(z_{n,i} \in B),$$

where $\chi(P) = 1$ if P is true and 0 if P is false. The following theorem is a specialization to the present case of the Banach-Alaoglu theorem, which asserts that the unit ball is compact in the weak* topology.

Theorem 1. *Given a sequence of measures $\{\mu_n \in M(E)\}$, there exists a subseqeunce, $\{\mu_{n_k}\}$, and a measure $\mu \in M(E)$ such that $\mu_{n_k} \to \mu$.* $\square$

Definition 2. A sequence of elementary measures will be called *regular* iff $\mu_n \to \mu$. The measure, μ, will be called the *interpolation measure* for the triangular interpolation scheme, Z.

Example 1. The following interpolation scheme does not give rise to a regular sequence of elementary measures.

$$\begin{array}{ccccccccccccc}
1 & & & & & & & & & & & & \\
1 & 0 & & & & & & & & & & & \\
1 & 0 & 1 & & & & & & & & & & \\
1 & 0 & 1 & 1 & & & & & & & & & \\
1 & 0 & 1 & 1 & 0 & & & & & & & & \\
1 & 0 & 1 & 1 & 0 & 0 & & & & & & & \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & & & & & & \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & & & & & \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & & & & \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & & & \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Clearly there is one subsequence of measures converging to a measure with mass $\frac{1}{2}$ at 0 and mass $\frac{1}{2}$ at 1, and another subsequence converging to a measure which has mass $\frac{3}{4}$ at 1 and $\frac{1}{4}$ at 0.

Since not every triangular interpolation scheme gives rise to a regular sequence of elementary measures, the question of verifying regularity is of some interest. In general, given a concrete sequence of elementary measures, a direct verification of regularity is quite straightforward. However, the following theorem, an elementary version of the Helley–Bray theorem, is particularly useful in many applications. Moreover, this theorem lends credence to the view that interpolation schemes which arise in practical problems can be expected to be regular.

THEOREM 2. *Given $\mu \in M(E)$ and $\{\mu_n \in M(E): n = 0, 1,...\}$, if for every open disk B, $\lim_{n\to\infty} \mu_n(B) = \mu(B)$, then $\mu_n \to \mu$.* □

The following example shows that the hypothesis of Theorem 2 is not necessary.

EXAMPLE 2. We define μ_n to have mass $1/n$ at the points $\{((n-1)/n)\, \theta^k : k = 0,..., n-1\}$, where θ is the primitive nth root of unity. It is a straightforward verification of the definition to show that this sequence is indeed regular. The limit measure has its support on the unit circle $|z| = 1$ and assigns to an arc of the circle the measure $1/(2\pi)$ times the arc length. However for the open unit disk, B, $\mu_n(B) = 1$ while $\mu(B) = 0$.

At this stage we have introduced the concept of an interpolation measure and have indicated that such a measure can be expected to be uniquely defined by each triangular interpolation scheme of practical interest. We will now indicate the relationship between the interpolation measure and questions of convergence for interpolating rational functions.

DEFINITION 3. Let Z be a triangular interpolation scheme, let E be the smallest closed set containing the points of Z, and let $f(z)$ be a function holomorphic on E. The *Hermite Interpolation Problem of Type* (m, n) is to find a rational function $r_{mn}(z)$ such that

$$r_{mn}(z) - f(z) = (z - z_{m+n,0})^{[m+n+1]}\, g(z),$$

where

$$(z - z_{m+n,0})^{[m+n+1]} \equiv (z - z_{m+n,0})(z - z_{m+n,1}) \cdots (z - z_{m+n,m+n}),$$

and $g(z)$ is analytic on an open set containing the points $\{z_{m+n,0}, z_{m+n,1},..., z_{m+n,m+n}\}$. The Hermite Interpolation Problem of Type (m, n) is not solvable in general. However, we can always find polynomials $p_{mn}(z)$ of degree at most m and $q_{mn}(z)$ of degree at most n such that

$$p_{mn}(z) - q_{mn}(z) f(z) = (z - z_{m+n,0})^{[m+n+1]} g(z). \tag{2.1}$$

The set of polynomials which satisfy (2.1) are equivalent rational forms and define a unique rational function, $r_{mn}(z)$, called the *rational interpolant.* When the Hermite Interpolation Problem of Type (m, n) is solvable, the rational interpolant is in fact the unique solution. A detailed discussion of these elementary facts is contained in [9].

The key item in questions of convergence is the polynomial $(z - z_{m+n,0})^{[m+n+1]}$, and the following observation links this polynomial to the interpolation measure.

$$|(z - z_{n,0})^{[n+1]}|^{1/(n+1)} = \exp\left(- \int_E \log(1/|z - \zeta|)\, d\mu_n(\zeta)\right).$$

Thus questions of convergence lead naturally to questions concerning the interpolation measure and its logarithmic potential (the integral in the exponent above). We next summarize some facts concerning logarithmic potentials. The proofs of these results may be found in [2, 4, 7, 9].

DEFINITION 4. Let E be a compact subset of the complex plane, and let $\mu \in M(E)$. Then the *logarithmic potential*, $u(z; \mu)$, is the function

$$u(z; \mu) \equiv \int_E \log(1/|z - \zeta|)\, d\mu(\zeta).$$

THEOREM 3. *The logarithmic potential, $u(z; \mu)$, is a harminic function on E^c and is lower semicontinuous and superharmonic on E.* □

3. THE MAJORIZATION THEOREM

In this section we will establish the key majorizing property of the logarithmic potential of an interpolation measure.

Notation. Let $N > 0$,

$$u_N(z; \mu) \equiv \int_E \min(\log(1/|z - \zeta|), N)\, d\mu(\zeta).$$

LEMMA. *Given a collection of normalized positive Borel measures, $\{\mu_\alpha \in M(E)\}$, the family of functions $\{u_N(z; \mu_\alpha)\}$ is equicontinuous.*

Proof. It is elementary to verify that given $\epsilon > 0$ the function min $(\log(1/|\xi|), N)$ is uniformly continuous with $\delta \leqslant \epsilon e^{-N}$. Hence, if $|z_0 - z_1| < \delta$, then

$$|\min(\log(1/|z_0 - \zeta|), N) - \min(\log(1/|z_1 - \zeta|), N)| < \epsilon,$$

for all ζ. But $\mu_\alpha(E) = 1$. Thus if $| z_0 - z_1 | < \delta$, then

$$| u_N(z_0; \mu_\alpha) - u_N(z_1; \mu_\alpha)| < \epsilon,$$

for every α. $\square$

THEOREM 4. *Let $\{\mu_n \in M(E)\}$ be a regular sequence of elementary measures with $\mu_n \to \mu$. Then*

(a) $\liminf_{n\to\infty} u(z; \mu_n) \geqslant u(z; \mu)$, *uniformly on compact subsets of the complex plane*; *and*

(b) $\lim_{n\to\infty} u(z; \mu_n) = u(z; \mu)$, *uniformly on compact subsets of E^c.*

Proof. First, note that $u(z; \mu_n) \geqslant u_N(z; \mu_n)$. Second, note that the collection $\{u_N(z; \mu), u_N(z; \mu_\alpha)\}$ is equicontinuous by the Lemma. Thus given ϵ choose δ so that

$$| u_N(z; \mu_n) - u_N(z'; \mu_n)| < \epsilon/3$$

whenever $| z - z' | < \delta$. Then, since we are concerned with $z \in K$, K a compact set, pick a finite set of points $\{\zeta_0, \zeta_1, \ldots, \zeta_k\}$ so that the set of disks $\{B(\zeta_0, \delta), \ldots, B(\zeta_k, \delta)\}$ cover K. Then for each ζ_i (since $\mu_n \to \mu$), we can choose N_i so that

$$| u_N(\zeta_i ; \mu) - u_N(\zeta_i ; \mu_n)| < \epsilon/3,$$

for every $n \geqslant N_i$. Then given any $z \in K$ there is a closest ζ_i, call it ζ^*, for which we will have

$$\begin{aligned} &| u_N(z; \mu) - u_N(z; \mu_n)| \\ &\quad = | u_N(z; \mu) - u_N(\zeta^*; \mu) + u_N(\zeta^*; \mu) - u_N(\zeta^*; \mu_n) + u_N(\zeta^*; \mu_n) \\ &\qquad - u_N(z; \mu_n)| < (\epsilon/3) + (\epsilon/3) + (\epsilon/3) = \epsilon, \end{aligned}$$

for every $n \geqslant N^* \equiv \max(N_0, \ldots, N_k)$. Thus

$$u(z; \mu_n) \geqslant u_N(z; \mu_n) > u_N(z; \mu) - \epsilon,$$

for every $n \geqslant N^*$. But N is arbitrary, so we must have

$$u(z; \mu_n) \geqslant u(z; \mu) - \epsilon,$$

for all $z \in K$ and for all $n \geqslant N^*$. This establishes (a).

For K a compact subset of E^c, we can find two points $\zeta_0 \in E$, $\zeta_1 \in K$, such that $|\zeta_0 - \zeta_1| = \text{dist}(E, K)$. Thus for $z \in K$,

$$u(z; \mu_n) \leqslant \log(1/|\zeta_0 - \zeta_1|).$$

Choosing $N > \log(1/|\zeta_0 - \zeta_1|)$ we have, for $z \in K$, $u_N(z; \mu_n) = u(z; \mu_n)$. Hence by the Lemma and the preceding argument we see that

$$|u(z; \mu) - u(z; \mu_n)| < \epsilon$$

for all $z \in K$, and for all $n \geqslant N^*$. This establishes (b). □

4. Convergence of Interpolating Rational Functions with a Fixed Number of Poles

In this section we present a theorem which describes conditions under which a sequence of rational interpolants with a fixed number of poles will converge. This result is a substantial, but entirely straightforward generalization of the recent theorem by Saff [6]. The underlying conditions are the following: Z is a bounded triangular interpolation scheme, whose associated sequence of elementary measures is regular, $\mu_n \to \mu$; E is the smallest closed set containing the points of Z; and $E_\rho \equiv \{z\colon e^{-u(z;\mu)} < \rho\}$.

THEOREM 5. *If $f(z)$ is holomorphic on E and meromorphic on E_ρ with precisely n poles, $\{\pi_1, ..., \pi_n\}$, inside E_ρ, then for all m sufficiently large, there exist rational functions, $r_{mn}(z) = p_{mn}(z)/q_{mn}(z)$, which solve the Hermite Interpolation Problem of Type (m, n) for $f(z)$ on Z. Moreover, the denominators, $q_{mn}(z)$, converge to $(z - \pi_1)^{[n]}$, uniformly on compact subsets of the complex plane, and the rational functions, $r_{mn}(z)$, converge to $f(z)$ uniformly on compact subsets of $E_\rho - \{\pi_1, ..., \pi_n\}$ as $m \to \infty$.*

Remark. It is important to note that in general E_ρ does not contain E.

Proof. The proof follows Saff's quite closely. Let $r_{mn}(z) = p_{mn}(z)/q_{mn}(z)$ and express $q_{mn}(z)$ as

$$q_{mn}(z) = \sum_{k=0}^{n} a_{mk}(z - \pi_1)^{[k]}$$

with $a_{mn} \equiv 1$. Let $h(z) \equiv q_{mn}(z)(z - \pi_1)^{[n]} f(z)$. Then $h(z)$ is holomorphic on an open set $\Omega \supset E \cup \bar{E}_\rho$. For any choice of the coefficients $\{a_{m0}, ..., a_{m,n-1}\}$, we can find a polynomial, $\hat{p}_{m+n,0}(z)$, which interpolates $h(z)$ at the points $\{z_{m+n,0}, \ldots, z_{m+n,m+n}\}$. Now by the Hermite integral representation of the interpolating polynomial, we have

$$\begin{aligned}\hat{p}_{m+n,0}(z) &= \frac{1}{2\pi i}\int_\Gamma \left[1 - \frac{(z - z_{m+n,0})^{[m+n+1]}}{(\zeta - z_{m+n,0})^{[m+n+1]}}\right] \frac{h(\zeta)}{(\zeta - z)}\, d\zeta \\ &= \sum_{k=0}^{n} \frac{a_{mk}}{2\pi i}\int_\Gamma \left[1 - \frac{(z - z_{m+n,0})^{[m+n+1]}}{(\zeta - z_{m+n,0})^{[m+n+1]}}\right] \frac{(\zeta - \pi_1)^{[k]}(\zeta - \pi_1)^{[n]} f(\zeta)}{(\zeta - z)}\, d\zeta \\ &= \sum_{k=0}^{n} a_{mk} C_k^{(m)}(z).\end{aligned}$$

Γ is a cycle contained in Ω homologous to 0, with winding number, $n(z, \Gamma) = 1$, for all $z \in E \cup \bar{E}_\rho$. Since $e^{-u(z;\mu)}$ is upper-semicontinuous, it assumes its maximum on compact sets. Thus for K a compact subset of E_ρ, let $\alpha \equiv \max\{e^{-u(z;\mu)}: z \in K\}$. Moreover, by definition, for all $z \in E_\rho{}^c$, $e^{-u(z;\mu)} \geqslant \rho$ and $\rho > \alpha$. Now,

$$\left| \frac{(z - z_{m+n,0})^{[m+n+1]}}{(\zeta - z_{m+n,0})^{[m+n+1]}} \right| = \exp(-(m+n+1)[u(z; \mu_{m+n}) - u(\zeta; \mu_{m+n})]).$$

By Theorem 4 we can find N so that if $m \geqslant N$ then $u(z; \mu_{m+n}) > u(z; \mu) - \epsilon$ for all $z \in K$, and $u(\zeta; \mu_{m+n}) < u(\zeta; \mu) + \epsilon$ for all $\zeta \in \Gamma$. Hence,

$$\begin{aligned}\left| \frac{(z - z_{m+n,0})^{[m+n+1]}}{(\zeta - z_{m+n,0})^{[m+n+1]}} \right| &\leqslant \exp(-(m+n+1)[u(z; \mu) - u(\zeta; \mu) - 2\epsilon]) \\ &\leqslant [(\alpha/\rho)\, e^{2\epsilon}]^{(m+n+1)},\end{aligned}$$

and we can choose ϵ sufficiently small so that $\alpha e^{2\epsilon}/\rho < 1$. Thus,

$$\lim_{m\to\infty} C_k^{(m)}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(\zeta - \pi_1)^{[k]}(\zeta - \pi_1)^{[n]} f(\zeta)}{(\zeta - z)}\, d\zeta$$

uniformly on compact subsets of E_ρ. Moreover by Weierstrass's theorem we also have

$$\lim_{m\to\infty} D^j C_k^{(m)}(z) = \frac{j!}{2\pi i}\int_\Gamma \frac{(\zeta - \pi_1)^{[k]}(\zeta - \pi_1)^{[n]} f(z)}{(\zeta - z)^{[j+1]}}\, d\zeta.$$

Now let $\gamma_i \equiv \sum_{j=0}^{i-1} \chi(\pi_j = \pi_i)$ and define $c_{ik}^{(m)} \equiv D^{\gamma_i} C_k^{(m)}(\pi_i)$. Then $(z - \pi_1)^{[n]}$ divides $\hat{p}_{m+n,0}(z)$ iff the linear system

$$\begin{bmatrix} c_{10}^{(m)} & \cdots & c_{1,n-1}^{(m)} \\ \vdots & & \vdots \\ c_{n0}^{(m)} & \cdots & c_{n,n-1}^{(m)} \end{bmatrix} \begin{bmatrix} a_{m0} \\ \vdots \\ a_{m,n-1} \end{bmatrix} = - \begin{bmatrix} c_{1n}^{(m)} \\ \vdots \\ c_{nn}^{(m)} \end{bmatrix}$$

is solvable. But Cauchy's Integral Theorem implies that $\lim_{m\to\infty} c_{ik}^{(m)}$ is zero for $k > i - 1$ and nonzero for $k = i - 1$. Thus

$$\lim_{m\to\infty} \det \begin{bmatrix} c_{10}^{(m)} & \cdots & c_{1,n-1}^{(m)} \\ \vdots & & \vdots \\ c_{n0}^{(m)} & \cdots & c_{n,n-1}^{(m)} \end{bmatrix} \neq 0.$$

Hence for m sufficiently large the linear system is uniquely solvable.

Let $q_{mn}(z)$ be determined so that $(z - \pi_1)^{[n]}$ divides $\hat{p}_{m+n,0}(z)$. Let $p_{mn}(z) \equiv \hat{p}_{m+n,0}(z)/(z - \pi_1)^{[n]}$. Then

$$p_{mn}(z) - q_{mn}(z) f(z) = (z - z_{m+n,0})^{[m+n+1]}\, g(z),$$

where $g(z)$ is holomorphic on E. Hence, $r_{mn}(z) \equiv p_{mn}(z)/q_{mn}(z)$ is the unique rational interpolant. However, since $\lim_{m\to\infty} c^{(m)} = 0$ for $i = 1,..., n$, and since $a_{mk} \to 0$ for $k = 0,..., n - 1$, we see that

$$\lim_{m\to\infty} q_{mn}(z) = (z - \pi_i)^{[n]},$$

uniformly on compact subsets of the complex plane. Since the poles are some finite distance from the set E, for m sufficiently large $q_{mn}(z)$ does not vanish on E and the rational interpolant, $r_{mn}(z)$, actually solves the Hermite Interpolation Problem for $f(z)$ on Z.

Thus, since K is compact, for sufficiently large m $| q_{mn}(z)(z - \pi_1)^{[n]} |$ is uniformly bounded below by a positive constant η. Hence, letting $M \equiv \max\{| h(\zeta)| : \zeta \in \Gamma\}$, and $\delta \equiv \operatorname{dist}(K, \Gamma)$, we see that for all $z \in K$,

$$| \hat{p}_{m+n,0}(z) - q_{mn}(z)(z - \pi_1)^{[n]} f(z)| \leqslant (M \mid \Gamma \mid /2\pi\delta)[(\alpha/\rho)\, e^{2\epsilon}]^{(m+n+1)}.$$

Consequently,

$$| r_{mn}(z) - f(z)| \leqslant (M \mid \Gamma \mid /2\pi\delta\eta)[(\alpha/\rho)\, e^{2\epsilon}]^{(m+n+1)}.$$

Thus $\lim_{m\to\infty} r_{mn}(z) = f(z)$ uniformly on compact subsets of $E_\rho - \{\pi_1 ,..., \pi_n\}$. Finally note that $\lim_{m\to\infty} p_{mn}(z) = (z - \pi_1)^{[n]} f(z)$ uniformly in a neighborhood of each π_i, and hence for m sufficiently large $p_{mn}(z)$ does not vanish at $\{\pi_1 ,..., \pi_n\}$. So for m sufficiently large $r_{mn}(z)$ has precisely n poles, counting multiplicities, which tend to the poles $\{\pi_1 ,..., \pi_n\}$ of $f(z)$. $\square$

5. Runge's Problem

About the turn of the century, Runge posed the following problem. Let $f(x) \equiv 1/(1 + x^2)$, and let $p_n(x)$ be the polynomial which interpolates $f(x)$ at

$n+1$ points uniformly distributed on the interval $[-5, 5]$. Does $p_n(x) \to f(x)$? For this problem we see that $\mu_n \to dt/10$ and

$$u(z; dt/10) = -(1/10)[-10 + (5-x)\log|5-z| + (5+x)\log|5+z| + y(\arg(5-z) - \arg(-5-z))],$$

where $z = x + iy$. In Fig. 1 we show some of the isopotential curves for $u(z; dt/10)$. In particular, the curve with value 2.46879067... passes through the poles of $f(x)$ at $\pm i$ and intersects the real line at ± 3.6333843.... Hence, the sequence $\{p_n(z)\}$ converges inside this curve. Moreover, the Majorization Theorem can be used to show that the sequence diverges outside this curve with the possible exception of points on the interval itself. Runge himself showed that the sequence diverges for $|x| > 3.633$..., and $x \neq 5$. The fact that our results do not assert anything concerning divergence on the interval itself indicates an area where the Majorization Theorem can possibly be refined.

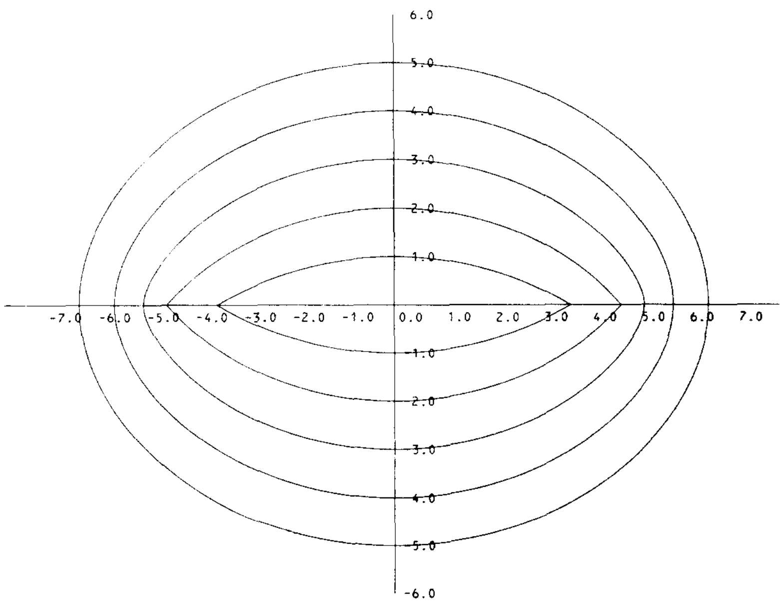

FIG. 1. Isopotential curves for the uniform distribution.

The hypothesis of Saff's theorem is stated in terms of the Green's function for the set, F, of limit points of the interpolation scheme. This hypothesis is equivalent to requiring that $\mu_n \to \nu$, where ν is the unique equilibrium distribution for the set F. In this setting, Walsh's results on maximal convergence [8] appear to be intimately related to Frostman's theorem [7,

p. 60; 9, p. 123]. It is well known, see, for example, Tsuji [7, p. 73] or [9, p. 133], that the zeros of the Tchebycheff polynomials yield an interpolation scheme with this property. For the interval $[-5, 5]$, $\mu_n \to \nu$, where

$$\nu \equiv dt/\pi(25 - t^2)^{1/2},$$

and

$$u(z; \nu) = \log(2/|\, z + (z^2 - 25)^{1/2} \,|).$$

It is easily verified that for $x \in [-5, 5]$, $e^{-u(x;\nu)} = 2.5$, and the isopotential curves for values larger than 2.5 are ellipses with foci at ± 5. Figure 2 illustrates some of these curves. Consequently, we see that for this scheme, the corresponding sequence of interpolating polynomials will converge to any function which is analytic on $[-5, 5]$.

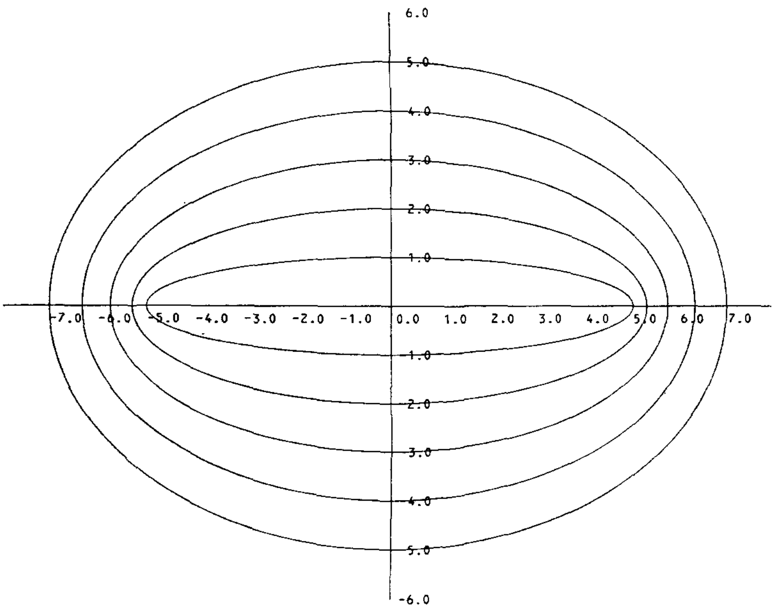

FIG. 2. Isopotential curves for the Tchebycheff distribution.

ACKNOWLEDGMENT

I wish to express my sincere thanks to Professor William B. Gragg for his advice and encouragement.

References

1. P. J. Davis, "Interpolation and Approximation," Blaisdell, 1963.
2. G. M. Goluzin, "Geometric Theory of Functions of a Complex Variable," Translations of Mathematical Monographs, Vol. 26, Amer. Math. Soc., Providence, R.I., 1969.
3. V. L. Gontscharoff, "Theory of Interpolation and Approximation," Moscow, 1954 (in Russian).
4. E. Hille, "Analytic Function Theory," Vol. II, Ginn/Blaisdell, Waltham, Mass., 1962.
5. V. I. Krylov, "Approximate Calculation of Integrals," Macmillan, New York, 1962.
6. E. B. Saff, An extension of Montessus de Ballore's theorem on the convergence of interpolating rational functions, *J. Approximation Theory* **6** (1972), 63–67.
7. M. Tsuji, "Potential Theory in Modern Function Theory," Maruzen, Tokyo, 1959.
8. J. L. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Domain," Colloquium Publications, Vol. XX, Amer. Math. Soc., Providence, R.I., 1969.
9. D. D. Warner, Hermite interpolation with rational functions, Doctoral Dissertation, University of California at San Diego, 1974.